International Journal of Theoretical Physics, Vol. 15, No. 9 (1976), pp. 677–689

Spontaneous Symmetry Breaking with Quaternionic Scalar Fields and Electron–Muon Mass Ratio

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Received: 29 September 1975

Abstract

We investigate the problem of using quaternionic scalar fields as Higg's mesons in theories of spontaneously broken symmetries. We are led to the symplectic $Sp(1, Q) \otimes U(1)$ as a possible gauge group for a unified theory of electromagnetic and weak interactions. The features of this model are worked out and compared with those of Weinberg's $SU(2) \otimes U(1)$ model.

1. *Introduction*

Analogous to the real and complex scalar fields which obey the usual Klein-Gordon equation, Finkelstein *et al.* (1962, 1963) have considered the desirability of employing quaternionic scalar wave functions and fields in reformulations of quantum mechanics and field theory. The elegance achieved encourages one to exploit their ideas, in order to construct viable theories of spontaneous symmetry breaking in which one expects to obtain a more economical unified description of electromagnetic and weak interactions.

Thus consider the real and complex free scalar fields $\varphi(x)$, which obey the Klein-Gordon equation

$$(\Box + m^2)\varphi(x) = 0$$

and have the following Lagrangians:

$$\mathscr{L} = \tfrac{1}{2}\{(\partial_\mu\varphi)^2 - m^2\varphi^2\}$$

and

$$\mathscr{L} = \partial_\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi$$

respectively, where

$$\varphi_1 = \frac{1}{\sqrt{2}}(\varphi + \varphi^\dagger), \qquad \varphi_2 = \frac{-i}{\sqrt{2}}(\varphi - \varphi^\dagger)$$

One could consider these fields in self-interaction and include a φ^4 term. For the complex scalar field we write

$$\mathscr{L} = \partial_\mu \varphi^\dagger \partial_\mu \varphi - V(\varphi^\dagger \varphi) \tag{1.1}$$

Where

$$V(\varphi^\dagger \varphi) = \mu^2 \varphi^\dagger \varphi + \lambda (\varphi^\dagger \varphi)^2$$

It is a well-known elementary result that this Lagrangian is invariant under the $U(1)$ transformation:

$$\varphi(x) \to e^{-i\alpha} \varphi(x), \qquad \alpha \neq \alpha(x)$$

If $\alpha = \alpha(x)$, invariance may be restored using the new Lagrangian

$$\mathscr{L} = (D_\mu \varphi)^\dagger (D_\mu \varphi) - V(\varphi^\dagger \varphi) \tag{1.2}$$

where

$$D_\mu = \partial_\mu - ieA_\mu$$

and A_μ is a gauge field that is arbitrary up to the transformation

$$A_\mu \to A_\mu - (1/e)\partial_\mu \alpha(x)$$

A free massless gauge field Lagrangian of the form

$$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu}$$

may be added to equation (1.2), where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Now in the thoroughly studied process of spontaneous symmetry breaking, if we set $\partial V/\partial\varphi = 0$ and look for nontrivial solutions, we get for $\mu^2 < 0$ and $\lambda > 0$

$$\varphi^\dagger \varphi = \varphi_1{}^2 + \varphi_2{}^2 = -\mu^2/\lambda = v^2$$

which can be satisfied by the arbitrary choice

$$\varphi_1 = v, \qquad \varphi_2 = 0$$

Quantum mechanically one writes

$$\langle 0 | \varphi_1(x) | 0 \rangle = v, \quad \langle 0 | \varphi_2(x) | 0 \rangle = 0$$

The fields φ_i may be redefined by writing

$$\chi_1 = \varphi_1 - v, \qquad \chi_2 = \varphi_2 \tag{1.3}$$

so that

$$\langle 0 | \chi_1(x) | 0 \rangle = \langle 0 | \chi_2(x) | 0 \rangle = 0$$

The popular Kibble–Higgs mechanism, so skilfully exploited in the construction of the Weinberg prototype gauge models, is that if we substitute equation

(1.3) into equation (1.1) we end up having one massless spin-zero particle χ_2, which is the Goldstone boson. On the other hand, if the substitution is into equation (1.2) rather than equation (1.1), we find that the Goldstone boson is absent. In its place, a massive gauge field appears. This result is readily obtained if instead of using equation (1.3) we parametrize $\varphi(x)$ in the so-called unitary gauge

$$\varphi(x) = (1/\sqrt{2})[\chi_1(x) + v] \exp[-i\chi_2(x)] \tag{1.4}$$

and then substitute (1.4) into equations (1.1) and (1.2).

2. *Quaternionic Higg's Meson*

In place of the real or complex scalar field, we could introduce the following quaternionic scalar field $\varphi(x)$ defined by

$$\varphi(x) = e_0\varphi_0(x) + e_1\varphi_1(x) + e_2\varphi_2(x) + e_3\varphi_3(x) \tag{2.1}$$

where the quaternion units e_i obey the usual relations

$$e_i^2 = -e_0, \quad e_ie_0 = e_0e_i = e_i, \quad i = 1, 2, 3$$

Also

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2$$

The component fields φ_i, ($i = 0, 1, 2, 3$) may be real or complex scalar fields. We shall consider only the case of real components in this section. The quaternionic conjugate field $\varphi^c(x)$ is defined by

$$\varphi^c(x) = e_0\varphi_0(x) - e_1\varphi_1(x) - e_2\varphi_2(x) - e_3\varphi_3(x) \tag{2.1a}$$

It is known that the quaternionic units can be represented by Pauli matrices as follows:

$$e_0 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 \to \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

This enables us to calculate where necessary, with the following matrix form of a quaternion:

$$\varphi(x) = \begin{pmatrix} X_1 & -X_2 \\ X_2^* & X_1^* \end{pmatrix}, \quad \varphi^c(x) = \begin{pmatrix} X_1^* & +X_2 \\ -X_2^* & X_1 \end{pmatrix}$$

Where $X_1 = \varphi_0 + i\varphi_1$ and $X_2 = \varphi_2 + i\varphi_3$; X_1^* and X_2^* are complex conjugates. We will evaluate traces at appropriate places using, for example,

$$\tfrac{1}{2}\mathrm{Tr}(\varphi^c\varphi) = X_1^*X_1 + X_2^*X_2$$

Obviously, the quaternionic scalar field obeys the Klein–Gordon equation. The free-field Lagrangian can be written as

$$\mathscr{L} = \mathrm{Tr}(\partial_\mu\varphi^c\partial_\mu\varphi) - \mu^2\mathrm{Tr}(\varphi^c\varphi)$$

We could include a self-interaction term and write

$$\mathscr{L} = \mathrm{Tr}(\partial_\mu \varphi^c \partial_\mu \varphi) - V(\varphi^c \varphi)$$

where

$$V(\varphi^c\varphi) = \mu^2 \mathrm{Tr}(\varphi^c\varphi) + \lambda \mathrm{Tr}(\varphi^c\varphi)^2 \tag{2.2}$$

This Lagrangian (2.2) is seen to be invariant under the symplectic $Sp(1, Q)$ transformation (Gourdin, 1967)

$$\begin{aligned} \varphi(x) &\to e^{-\beta}\varphi(x) \\ \varphi^c(x) &\to \varphi^c e^{+\beta} \end{aligned} \tag{2.3}$$

where

$$\beta = e_1\alpha_1 + e_2\alpha_2 + e_3\alpha_3$$

Here the $e_i (i = 1, 2, 3)$ are again the quaternion units. Equation (2.3) is a simple generalization of the phase transformation $\varphi \to e^{-i\alpha}\varphi$ for a complex scalar field. The parameters α_i are real c numbers with $\alpha_i \neq \alpha_i(x)$.

If we let the parameters α_i become functions of x so that $\beta = \beta(x)$, we could restore invariance of equation (2.2) under (2.3), by the usual minimal substitution principle. We replace equation (2.2) by

$$\mathscr{L} = (D_\mu\varphi)^c \cdot (D_\mu\varphi) - V(\varphi^c\varphi) \tag{2.4}$$

where

$$D_\mu = \partial_\mu - gQ_\mu$$

with

$$Q_\mu = \sum_{i=1}^{3} e_i A_\mu{}^i$$

We have the gauge freedom

$$A_\mu{}^i \to A_\mu{}^i - (1/g)\partial_\mu \alpha^i(x)$$

or

$$Q_\mu \to Q_\mu - (1/g)\partial_\mu \beta(x) \tag{2.5}$$

The new Lagrangian is

$$\begin{aligned} \mathscr{L} &= [(\partial_\mu - gQ_\mu)\varphi]^c \cdot [(\partial_\mu - gQ)\varphi_\mu] - V(\varphi^c\varphi) \\ &= \partial_\mu\varphi^c\partial_\mu\varphi - g\varphi^c Q_\mu{}^c \partial_\mu\varphi - g(\partial_\mu\varphi^c)Q_\mu\varphi + g^2\varphi^c Q_\mu{}^c Q_\mu \varphi - V(\varphi^c\varphi) \end{aligned} \tag{2.6}$$

One can verify that this Lagrangian (2.6) is invariant under local gauge $Sp(1, Q)$ transformations. A generalization of well-known results (Abers and Lee, 1973) for the transformation law of the gauge field $(\boldsymbol{\tau} \cdot \mathbf{A})$ under a non-

Abelian local gauge group G, leads to the following transformation law for Q_μ under local gauge $Sp(1, Q)$:

$$Q_\mu \to Q'_\mu = e^{-\beta} Q_\mu e^{\beta} - (1/g) e^{-\beta} (\partial_\mu \beta) e^{\beta} \tag{2.7}$$

Taking the quaternionic conjugate, we write

$$Q_\mu{}^c \to (Q_\mu{}^c)' = e^{-\beta} Q_\mu{}^c e^{\beta} + (1/g) e^{-\beta} (\partial_\mu \beta) e^{\beta} \tag{2.8}$$

One can now check that equation (2.6) is invariant under the set of local gauge transformations (2.3) and (2.7).

To the Lagrangian (2.6), we have to add the free field Lagrangian for the quaternionic vector field Q_μ. The form of $\mathscr{L}(Q_\mu)$ has to be chosen such that it is separately invariant under equation (2.7). After a lengthy algebra, one verifies that the following form can be chosen for $\mathscr{L}(Q_\mu)$:

$$\mathscr{L}(Q_\mu) = -\tfrac{1}{4} F_{\mu\nu} F^c_{\mu\nu} \tag{2.9}$$

where

$$F_{\mu\nu} = -(\partial_\mu Q_\nu - \partial_\nu Q_\mu) + g\,[Q_\mu, Q_\nu]_- + (1/g)\,[\partial_\mu \beta, \partial_\nu \beta]_-$$

$$F^c_{\mu\nu} = -(\partial_\mu Q_\nu{}^c - \partial_\nu Q_\mu{}^c) - g\,[Q_\mu{}^c, Q_\nu{}^c]_- - (1/g)\,[\partial_\mu \beta, \partial_\nu \beta]_-$$

(The last term in this equation may in fact be dropped, since it is of second order in the gauge transformation parameter β.)

Having now constructed the Lagrangians (2.6) and (2.9), which are separately invariant under local gauge $Sp(1, Q)$ transformations, we can consider breaking the symmetry spontaneously. We have

$$V(\varphi^c \varphi) = \mu^2 \varphi^c \varphi + \lambda (\varphi^c \varphi)^2$$

so that

$$\partial V / \partial \varphi = 0$$

implies

$$\varphi^c \varphi = -\mu^2 / \lambda$$

Requiring as usual that $\mu^2 < 0, \lambda > 0$, we can choose

$$\langle 0 | \varphi_0(x) | 0 \rangle = -\mu^2/\lambda = \eta$$

$$\langle 0 | \varphi_i(x) | 0 \rangle = 0, \qquad i = 1, 2, 3$$

We introduce the field $\chi(x)$ defined by

$$\chi(x) = \varphi_0(x) - \eta$$

such that

$$\langle 0 | \chi | 0 \rangle = 0$$

In the so-called unitary gauge, we can now parametrize the quaternion field $\varphi(x)$ by

$$\varphi(x) = \rho e^{-\xi(x)}$$

where

$$\xi(x) = \sum_{i=1}^{3} e_i \varphi_i(x)$$

and

$$\rho(x) = (1/\sqrt{2})[\chi(x) + \eta] \tag{2.10}$$

We can now choose a particular gauge in equation (2.8) such that the $\xi(x)$ field in equation (2.10) drops out. The net result is equivalent to simply substituting

$$\varphi(x) = \rho(x) = (1/\sqrt{2})[\chi(x) + \eta] \tag{2.11}$$

into the Lagrangian (2.6). We obtain the following result (after taking traces):

$$\begin{aligned}\mathscr{L} &= \partial_\mu \rho \partial_\mu \rho + 2g^2\rho^2 \{(A_\mu{}^1)^2 + (A_\mu{}^2 + iA_\mu{}^3)(A_\mu{}^2 - iA_\mu{}^3)\} \\ &\quad - \mu^2\rho^2 - \lambda\rho^4 + \mathscr{L}(Q_\mu) \\ &= \tfrac{1}{2}\partial_\mu\chi\partial_\mu\chi + g^2(\chi^2 + 2\eta\chi + \eta^2)[(W_\mu{}^0)^2 + W_\mu{}^+ W_\mu{}^-] \\ &\quad - \tfrac{1}{4}\lambda\chi^4 - \lambda\eta\chi^3 - \lambda\eta^2\chi^2 + \mathscr{L}(Q_\mu)\end{aligned} \tag{2.12}$$

We see that, as expected, three massive vector mesons $W_\mu{}^0$, $W_\mu{}^+$, and $W_\mu{}^-$ have emerged.

Further generalizations of these results are now possible. In place of the quaternionic scalar field $\varphi(x)$, one could use a generalized hypercomplex scalar field (Ndili and Chukwumah 1974). The manipulations as discussed above carry through, but the algebra is more tedious. We suppress the details.

3. *Quaternionic Spinor Field*

We could next introduce the quaternionic spinor field defined by

$$\psi(x) = e_0\psi_0(x) + e_1\psi_1(x) + e_2\psi_2(x) + e_3\psi_3(x)$$

with

$$\psi^c(x) = e_0\bar{\psi}_0(x) - e_1\bar{\psi}_1(x) - e_2\bar{\psi}_2(x) - e_3\bar{\psi}_3(x)$$

where the $\psi_i (i = 0, 1, 2, 3)$ are spinor fields.

In matrix form we write

$$\psi(x) = \begin{pmatrix} \psi_0 + i\psi_1 & -(\psi_2 + i\psi_3) \\ \psi_2 - i\psi_3 & \psi_0 - i\psi_1 \end{pmatrix}$$

$$\psi^{c(x)} = \begin{pmatrix} \bar{\psi}_0 - i\bar{\psi}_1 & \bar{\psi}_2 + i\bar{\psi}_3 \\ -(\bar{\psi}_2 - i\bar{\psi}_3) & \bar{\psi}_0 + i\bar{\psi}_1 \end{pmatrix}$$

The free massless Lagrangian for this quaternionic spinor field is given by

$$\mathscr{L} = \tfrac{1}{2}\mathrm{Tr}(\psi^c \gamma_\lambda \partial_\lambda \psi)$$
$$= \sum_{i=0}^{3} \bar{\psi}_i \gamma_\mu \partial_\mu \psi_i$$

This Lagrangian is invariant under the transformation

$$\psi \to e^{-\beta}\psi, \qquad \psi^c \to \psi^c e^{\beta}$$

for $\beta \neq \beta(x)$. For $\beta = \beta(x)$ we restore invariance as before by using the new Lagrangian

$$\mathscr{L} = \psi^c \gamma_\mu (\partial_\mu - gQ_\mu)\psi + \mathscr{L}(Q_\mu) \tag{3.1}$$

where the quaternionic gauge field Q_μ transforms as in equation (2.7). The Lagrangian (3.1) is invariant under local gauge $Sp(1, Q)$ transformations.

4. *Weinberg–Salam Model with Quaternionic Fields*

Suppose we now take the known leptons e, ν_e, μ, and ν_μ and form the left- and right-handed fields

$$e_L = \frac{1-\gamma_5}{2} e, \qquad \mu_L = \frac{1-\gamma_5}{2}\mu$$
$$e_R = \frac{1+\gamma_5}{2} e, \qquad \mu_R = \frac{1+\gamma_5}{2}\mu \tag{4.1}$$

Also we have

$$\nu_L{}^e = \frac{1-\gamma_5}{2}\nu_e, \qquad \nu_L{}^\mu = \frac{1-\gamma_5}{2}\nu_\mu$$

We could form all the four left-handed fields into a quaternionic spinor field:

$$L = e_0\psi_0 + e_1\psi_1 + e_2\psi_2 + e_3\psi_3 \tag{4.2}$$

where we could for example make the identification

$$\psi_0 = \mu_L, \qquad \psi_1 = e_L$$
$$\psi_2 = \nu_L{}^\mu, \qquad \psi_3 = \nu_L{}^e \tag{4.3}$$

Then we could write

$$L = \begin{pmatrix} E_L & -\nu_L \\ \nu_L^* & E_L^* \end{pmatrix} \tag{4.4}$$

where

$$E_L = \psi_0 + i\psi_1 = \mu_L + ie_L = \mu_L^- + ie_L^-$$
$$E_L^* = \psi_0 - i\psi_1 = \mu_L^* - ie_L^* = \mu_L^+ - ie_L^+$$
$$\nu_L = \psi_2 + i\psi_3 = \nu_L^{\mu} + i\nu_L^{e} = \nu_L^{\mu^-} + i\nu_L^{e^-}$$
$$\nu_L^* = \psi_2 - i\psi_3 = \nu_L^{*\mu} - i\nu_L^{*e} = \nu_L^{\mu^+} - i\nu_L^{e^+}$$

The quaternionic conjugate field becomes

$$L^c = \begin{pmatrix} \bar{E}_L & \bar{\nu}_L^* \\ -\bar{\nu}_L & \bar{E}_L^* \end{pmatrix} = \begin{pmatrix} \bar{\psi}_0 - i\bar{\psi}_1 & \bar{\psi}_2 + i\bar{\psi}_3 \\ -(\bar{\psi}_2 - i\bar{\psi}_3) & \bar{\psi}_0 + i\bar{\psi}_1 \end{pmatrix}$$

where

$$\begin{aligned} \bar{E}_L &= \bar{\mu}_L - i\bar{e}_L = \overline{\mu_L^-} - i\overline{e_L^-} \\ \overline{E_L^*} &= \overline{\mu_L^*} + i\overline{e_L^*} = \overline{\mu_L^+} + i\overline{e_L^+} \\ \overline{\nu_L} &= \overline{\nu_L^{\mu^-}} - i\overline{\nu_L^{e^-}} \\ \overline{\nu_L^*} &= \overline{\nu_L^{\mu^+}} + \overline{\nu_L^{e^+}} \end{aligned} \qquad (4.5)$$

We form also the field

$$E_R = \mu_R + ie_R = R$$

Then choosing our gauge group as the direct product group $Sp(1, Q) \otimes U(1)$, we classify the field L into the defining representation of $Sp(1, Q)$, while the right-handed field E_R we classify as a singlet under $U(1)$. Finally we also introduce a quaternionic scalar Higg's meson $\varphi(x)$ given by equations (2.1) and (2.1a).

The transformations of these fields under $Sp(1, Q) \otimes U(1)$ will then be as follows:

Under $Sp(1, Q)$:

$$L \to e^{-\beta}L$$
$$R \to R$$
$$\varphi \to e^{-\beta}\varphi$$

Under $U(1)$:

$$\begin{aligned} L &\to e^{i\Lambda}L \\ R &\to e^{i\Lambda}R \\ \varphi &\to \varphi \end{aligned} \qquad (4.6)$$

where

$$\beta = \beta(x) = \sum_{i=1}^{3} e_i\alpha_i(x)$$

and $\Lambda = \Lambda(x)$.

Working now with these fields L, R, and φ we write down the Lagrangian

$$\mathscr{L} = L^c\gamma_\lambda\partial_\lambda L + \bar{R}\gamma_\lambda\partial_\lambda R + \partial_\lambda\varphi^c\partial_\lambda\varphi - V(\varphi^c\varphi) + \mathscr{L}_{\text{int}} \tag{4.7}$$

where $\mathscr{L}_{\text{int}}$ is a suitable interaction term involving L, R, and φ in interaction. It is understood that in equations like (4.7), traces will be taken where appropriate in order to obtain the final Lagrangian in a suitable form.

To obtain a Lagrangian that is invariant under local gauge $Sp(1, Q) \otimes U(1)$ we again use the minimal substitution principle:

$$\begin{aligned} L^c\gamma_\lambda\partial_\lambda L &\to L^c\gamma_\lambda(\partial_\lambda - gQ_\lambda - g'B_\lambda)L \\ \bar{R}\gamma_\lambda\partial_\lambda R &\to \bar{R}\gamma_\lambda(\partial_\lambda - g'B_\lambda)R \\ \partial_\lambda\varphi^c\partial_\lambda\varphi &\to [\partial_\lambda\varphi^c + g\varphi^c Q_\lambda + ig'B_\lambda\varphi^c]\,[\partial_\lambda\varphi - gQ_\lambda\varphi - ig'B_\lambda\varphi] \end{aligned} \tag{4.8}$$

where B_λ and Q_λ are two gauge fields associated with $U(1)$ and $Sp(1, Q)$, respectively. We have

$$Q = \sum_{i=1}^{3} e_i A_\lambda{}^i = \begin{pmatrix} iA_\lambda{}^1 & -(A_\lambda{}^2 + iA_\lambda{}^3) \\ A_\lambda{}^2 - iA_\lambda{}^3 & -iA_\lambda{}^1 \end{pmatrix} \tag{4.9}$$

These gauge fields transform as follows under $Sp(1, Q) \otimes U(1)$:
Under $Sp(1, Q) \otimes U(1)$:

$$\begin{aligned} &Q_\lambda \to Q'_\lambda = e^{-\beta}Q_\lambda e^\beta - (1/g)e^{-\beta}(\partial_\lambda\beta)e^\beta \\ &Q_\lambda{}^c \to (Q_\lambda{}^c)' = e^{-\beta}Q_\lambda{}^c e^\beta + (1/g)e^{-\beta}(\partial_\lambda\beta)e^\beta \\ &\beta_\lambda \to \beta_\lambda \end{aligned}$$

Under $U(1)$:

$$\begin{aligned} Q_\lambda &\to Q_\lambda \\ B_\lambda &\to B'_\lambda = B_\lambda - (1/g')\partial_\lambda\Lambda(x) \end{aligned} \tag{4.10}$$

We shall sometimes use the notation

$$B_\lambda{}^1 = iB_\lambda$$

The gauge-invariant Lagrangian becomes

$$\mathscr{L} = \mathscr{L}_{\text{lepton}} + \mathscr{L}_{\text{scalar}} + \mathscr{L}_{\text{int}} + \mathscr{L}(Q_\mu) + \mathscr{L}(B_\mu) \tag{4.11}$$

where

$$\begin{aligned} \mathscr{L}_{\text{lepton}} = \operatorname{Tr}\{&L^c\gamma_\lambda(\partial_\lambda - gQ_\lambda - g'B_\lambda)L\} \\ &+ 2\bar{R}\gamma_\lambda(\partial_\lambda - g'B_\lambda)R \end{aligned} \tag{4.12}$$

$$\begin{aligned} \mathscr{L}_{\text{scalar}} = \operatorname{Tr}\{&[\partial_\lambda\varphi^c + g\varphi^c Q_\lambda + ig'B_\lambda\varphi^c]\,[\partial_\lambda\varphi - gQ_\lambda\varphi - ig'B_\lambda\varphi]\} \\ &- V(\varphi^c\varphi) \end{aligned} \tag{4.13}$$

$$\mathscr{L}(B_\lambda) = -\tfrac{1}{4} f_{\lambda\nu} f_{\lambda\nu} \tag{4.14}$$

with

$$f_{\lambda\nu} = \partial_\lambda B_\nu - \partial_\nu B_\lambda$$

while $\mathscr{L}(Q_\mu)$ is given by equation (2.9).

We now break the $Sp(1, Q) \otimes U(1)$ symmetry sponaneously as in section 2, by substituting equation (2.11) into equation (4.11).

We get the following results:

$$\mathscr{L}_{\text{scalar}} = \partial_\lambda \rho \partial_\lambda \rho + 2\rho^2 \{g^2(A_\lambda^2 + iA_\lambda^3)(A_\lambda^2 - iA_\lambda^3) + (gA_\lambda^1 + ig'B_\lambda)(gA_\lambda^1 - ig'B_\lambda)\} - V(\varphi^c\varphi) \tag{4.15}$$

Suppose we now define the following fields:

$$W_\lambda^+ = A_\lambda^2 - iA_\lambda^3, \qquad W_\lambda^- = A_\lambda^2 + iA_\lambda^3$$

or

$$A_\lambda^2 = \tfrac{1}{2}(W_\lambda^+ + W_\lambda^-), \qquad iA_\lambda^3 = \tfrac{1}{2}(W_\lambda^- - W_\lambda^+)$$

$$Z_\lambda^0 = \frac{gA_\lambda^1 + g'B_\lambda^1}{\sqrt{g^2 + g'^2}} = A_\lambda^1 \cos\theta + B_\lambda^1 \sin\theta$$

$$\bar{Z}_\lambda^0 = \frac{gA_\lambda^1 - g'B_\lambda^1}{\sqrt{g^2 + g'^2}} = A_\lambda^1 \cos\theta - B_\lambda^1 \sin\theta$$

$$A_\lambda = \frac{g'A_\lambda^1 - gB_\lambda^1}{\sqrt{g^2 + g'^2}} = A_\lambda^1 \sin\theta - B_\lambda^1 \cos\theta$$

$$\bar{A}_\lambda = \frac{g'A_\lambda^1 + gB_\lambda^1}{\sqrt{g^2 + g'^2}} = A_\lambda^1 \sin\theta + B_\lambda^1 \cos\theta \tag{4.16}$$

or

$$A_\lambda^1 = Z_\lambda^0 \cos\theta + A_\lambda \sin\theta, \qquad B_\lambda^1 = iB_\lambda = Z_\lambda^0 \sin\theta - A_\lambda \cos\theta$$

where $\tan\theta = g'/g$.

Putting (4.16) and (2.11) into equation (4.15) we get finally

$$\mathscr{L}_{\text{scalar}} = \tfrac{1}{2}\partial_\lambda\chi\partial_\lambda\chi - \tfrac{1}{4}\lambda\chi^4 - \lambda\eta\chi^3 - \lambda\eta^2\chi^2 + (\chi + \eta)^2\{g^2 W_\lambda^+ W_\lambda^- + (g^2 + g'^2)Z_\lambda^0 Z_\lambda^{-0}\} \tag{4.17}$$

This shows that three vector mesons have acquired mass, while the fourth vector meson A_λ remains massless. The massless vector meson A_λ is expected to be identified with the photon.

Next we evaluate $\mathscr{L}_{\text{lepton}}$ given by equation (4.12).

We have

$$\mathscr{L}_{\text{lepton}} = \mathrm{Tr}(L^c\gamma_\lambda\partial_\lambda L) - g\,\mathrm{Tr}(L^c\gamma_\lambda Q_\lambda L) - g'B_\lambda\{\mathrm{Tr}(L^c\gamma_\lambda L) + 2\bar{R}\gamma_\lambda R\} + 2\bar{R}\gamma_\lambda\partial_\lambda R$$

Using equations (4.4), (4.5), (4.9), and (4.16) we get

$$\mathrm{Tr}(L^c\gamma_\lambda L) = \bar{E}_L\gamma_\lambda E_L + \bar{\nu}_L^*\gamma_\lambda \nu_L^* + \bar{\nu}_L\gamma_\lambda\nu_L + \bar{E}_L^*\gamma_\lambda E_L^*$$

$$\mathrm{Tr}(L^c\gamma_\lambda Q_\lambda L) = (\bar{E}_L\gamma_\lambda E_L - \bar{\nu}_L^*\gamma_\lambda\nu_L^* + \bar{\nu}_L\gamma_\lambda\nu_L - \bar{E}_L^*\gamma_\lambda E_L^*)iA_\lambda{}^1 + (\bar{\nu}_L\gamma_\lambda E_L^* - \bar{E}_L\gamma_\lambda\nu_L^*)W_\lambda{}^- + (\bar{\nu}_L^*\gamma_\lambda E_L - \bar{E}_L^*\gamma_\lambda\nu_L)W_\lambda{}^+$$

We then get that

$$\begin{aligned}\mathscr{L}_{\text{lepton}} = {} & \bar{E}_L\gamma_\lambda\partial_\lambda E_L + 2\bar{E}_R\gamma_\lambda\partial_\lambda E_R + \bar{E}_L^*\gamma_\lambda\partial_\lambda E_L^* + \bar{\nu}_L^*\gamma_\lambda\partial_\lambda\nu_L^* + \bar{\nu}_L\gamma_\lambda\partial_\lambda\nu_L \\ & + g(\bar{E}_L\gamma_\lambda\nu_L^* - \bar{\nu}_L\gamma_\lambda E_L^*)W_\lambda^- + g(\bar{E}_L^*\gamma_\lambda\nu_L - \bar{\nu}_L^*\gamma_\lambda E_L)W_\lambda{}^+ \\ & - \frac{2igg'A_\lambda}{\sqrt{g^2+g'^2}}(\bar{E}_L\gamma_\lambda E_L + \bar{E}_R\gamma_\lambda E_R + \nu_L\gamma_\lambda\nu_L) \\ & + \frac{iZ_\lambda{}^0}{\sqrt{g^2+g'^2}}\{g'^2(\bar{E}_L\gamma_\lambda E_L + 2\bar{E}_R\gamma_\lambda E_R + \bar{E}_L^*\gamma_\lambda E_L^* + \bar{\nu}_L^*\gamma_\lambda\nu_L^* + \bar{\nu}_L\gamma_\lambda\nu_L) \\ & - g^2(\bar{E}_L\gamma_\lambda E_L - \bar{E}_L^*\gamma_\lambda E_L^* + \bar{\nu}_L\gamma_\lambda\nu_L - \bar{\nu}_L^*\gamma_\lambda\nu_L^*)\}\end{aligned} \tag{4.18}$$

Finally substituting the notations of equations (4.4) and (4.5) into equation (4.18) (and dropping a factor of 2) we obtain

$$\begin{aligned}\mathscr{L}_{\text{lepton}} = {} & \bar{\mu}\gamma_\lambda\partial_\lambda\mu + \bar{e}\gamma_\lambda\partial_\lambda e - i\bar{e}\gamma_\lambda\frac{1+\gamma_5}{2}\partial_\lambda\mu + i\bar{\mu}\gamma_\lambda\frac{1+\gamma_5}{2}\partial_\lambda e \\ & + \bar{\nu}_\mu\gamma_\lambda\frac{1-\gamma_5}{2}\partial_\lambda\nu_\mu + \bar{\nu}_e\gamma_\lambda\frac{1-\gamma_5}{2}\partial_\lambda\nu_e \\ & + \frac{g}{2}W_\lambda^-\left\{\bar{\mu}\gamma_\lambda\frac{1-\gamma_5}{2}\nu_\mu - \bar{e}\gamma_\lambda\frac{1-\gamma_5}{2}\nu_e - i\bar{e}\gamma_\lambda\frac{1-\gamma_5}{2}\nu_\mu\right. \\ & - i\bar{\mu}\gamma_\lambda\frac{1-\gamma_5}{2}\nu_e - \bar{\nu}_\mu\gamma_\lambda\frac{1-\gamma_5}{2}\mu^+ + i\bar{\nu}_e\gamma_\lambda\frac{1-\gamma_5}{2}\mu^+ \\ & \left. + i\bar{\nu}_\mu\gamma_\lambda\frac{1-\gamma_5}{2}e^+ + \bar{\nu}_e\gamma_\lambda\frac{1-\gamma_5}{2}e^+\right\} \\ & + \frac{g}{2}W_\lambda{}^+\left\{\overline{\mu^+}\gamma_\lambda\frac{1-\gamma_5}{2}\nu_\mu - \overline{e^+}\gamma_\lambda\frac{1-\gamma_5}{2}\nu_e + i\overline{e^+}\gamma_\lambda\frac{1-\gamma_5}{2}\nu_\mu\right. \\ & + i\overline{\mu^+}\gamma_\lambda\frac{1-\gamma_5}{2}\nu_e - \bar{\nu}_\mu\gamma_\lambda\frac{1-\gamma_5}{2}\overline{\mu^+} + i\bar{\nu}_e\gamma_\lambda\frac{1-\gamma_5}{2}\mu^+ \\ & \left. - i\bar{\nu}_\mu\gamma_\lambda\frac{1-\gamma_5}{2}e + \bar{\nu}_e\gamma_\lambda\frac{1-\gamma_5}{2}e\right\} \\ & - \frac{igg'A_\lambda}{\sqrt{g^2+g'^2}}\{\bar{\mu}\gamma_\lambda\mu + \bar{e}\gamma_\lambda e + i\bar{\mu}\gamma_\lambda e - i\bar{e}\gamma_\lambda\mu + \bar{\nu}_L\gamma_\lambda\nu_L\}\end{aligned}$$

$$+\frac{iZ_\lambda^0}{\sqrt{g^2+g'^2}}\left\{g'^2\left(\bar{\mu}\gamma_\lambda\mu+\bar{e}\gamma_\lambda e+i\bar{\mu}\gamma_\lambda\frac{1+\gamma_5}{2}e-i\bar{e}\gamma_\lambda\frac{1+\gamma_5}{2}\mu\right.\right.$$

$$\left.+\bar{\nu}_\mu\gamma_\lambda\frac{1-\gamma_5}{2}\nu_\mu+\bar{\nu}_e\gamma_\lambda\frac{1-\gamma_5}{2}\nu_e\right)-g^2\left(i\bar{\mu}\gamma_\lambda\frac{1-\gamma_5}{2}e\right.$$

$$\left.\left.-i\bar{e}\gamma_\lambda\frac{1-\gamma_5}{2}\mu-i\bar{\nu}_e\gamma_\lambda\frac{1-\gamma_5}{2}\nu_\mu+i\bar{\nu}_\mu\gamma_\lambda\frac{1-\gamma_5}{2}\nu_e\right)\right\} \quad (4.19)$$

Next we write down $\mathscr{L}_{\text{int}}$. We choose

$$\mathscr{L}_{\text{int}}=f\{\text{Tr}(\bar{R}\varphi L)+\text{Tr}(L^c\varphi R)\}$$

where

$$\text{Tr}(\bar{R}\varphi L)=\rho(\bar{\psi}_0-i\bar{\psi}_1)\cdot(\psi_0-\gamma_5\psi_0)$$
$$\text{Tr}(L^c\varphi R)=\rho(\bar{\psi}_0+\gamma_5\bar{\psi}_0)\cdot(\psi_0+i\psi_1)$$

so that

$$\mathscr{L}_{\text{int}}=2f\rho\bar{\psi}_0\psi_0+if\rho\bar{\psi}_0(1+\gamma_5)\psi_1$$

That is,

$$\mathscr{L}_{\text{int}}=\sqrt{2}\eta f\bar{\mu}\mu+(i/\sqrt{2})f\bar{\mu}(1+\gamma_5)e\chi+\cdots \quad (4.20)$$

This equation would imply that as a result of the spontaneous symmetry breaking, the muon acquires a mass while the electron remains massless. Since the physical mass of the electron is nearly zero compared to that of the muon, the above result appears to be in the right direction. One may in fact interpret our model as a zeroth-order model in the sense of Weinberg (1972a, b); also Georgi and Glashow (1972, 1973). In this case, one considers that the small finite mass of the electron is generated by radiative processes that are superimposed on the zeroth-order lepton mass spectrum. The latter state of affairs is considered to arise as a direct consequence of the spontaneous symmetry breaking. However, it is still an open question how many loops one would need to consider in order to actually obtain the empirical electron–muon mass ratio

$$m_e/m_\mu\simeq\tfrac{2}{3}\alpha$$

where α is the fine-structure constant. This point needs closer study.

5. *Discussion*

Compared, however, to the original Weinberg–Salam model, (Weinberg, 1967 and 1971; Salam, 1967) it is obvious that our model incorporates electrons and muons in a more natural way. The muon terms, for example, are not added by hand as in the Weinberg–Salam model. Also while, in the Weinberg–Salam model, the electron–muon mass difference cannot arise directly from spontaneous symmetry breaking, but by an arbitrary choice of different coupling constants in the interaction terms, we find that, in our model, the process of spontaneous symmetry breaking leads directly to a massive muon and a massless electron. The quaternionic model is therefore a zeroth-order gauge model.

It is in fact the most economical zeroth-order gauge model, compared to other recently proposed zeroth-order gauge models for electron–muon mass ratio (Frenkel and Ebel 1973; Mohapatra 1974; Fritzsch and Minkowski 1974).

The quaternionic model is, however, not without its own problems. If we consider the total Lagrangian equation (4.11), which is obtained by adding the pieces (4.17), (4.19), (4.20), (4.14), and (2.9), we observe the appearance of a $\bar{\nu}\nu\gamma$ vertex. This term cannot be readily removed, although reasonable models of neutrino–photon vertex can be constructed. In addition to the problem of $\bar{\nu}\nu\gamma$ vertex, we have also that some terms in our Lagrangian violate the separate conservation laws of electron and muon numbers. This feature is perhaps not so objectionable in so far as the existing experimental tests are not yet sufficient to discriminate between various schemes (Marshak *et al.*, 1969) that have been proposed for conservation of lepton numbers. Finally, neutral current terms like $\bar{e}\gamma_\lambda\mu$ and $\bar{\mu}\gamma_\lambda e$, which are present in the photon term, must also be killed since they will otherwise lead to such unobserved processes as $\mu \to e + \gamma$. They would also predict significant muon exchange contributions to conventional Compton scattering. At the moment, we do not know how to refine the model in order to get over these difficulties.

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